comes, from Eq. (6),

$$\mathbf{h} = \{I_0\}\cdot\boldsymbol{\omega}_0 + \{I_1\}\cdot\boldsymbol{\omega}_1 + m_1(1 - m_1/M)\mathbf{r}_1 \times (d\mathbf{r}_1/dt) \quad (13)$$

Equations (12) and (13) show that we could calculate these kinetic energy and angular momentum as if the main body center of mass were the over-all satellite center of mass provided that we replace the auxiliary body mass m_1 by a reduced mass $m_1(1 - m_1/M)$. As an application of practical interest, we can now apply this result to assess the accuracy of an unsymmetrical yo-yo despin device (see Fig. 1) described in Ref. 3. The length of the yo-yo cord required for reducing the satellite spin to zero is given in Ref. 3 as

$$l_f = (a^2 + I_0/m_1)^{1/2} \quad (14)$$

the derivation being based on the conservation of energy and angular momentum and on the approximation that there is negligible difference between the satellite and system (satellite plus yo-yo) centers of mass. Of course, for the yo-yo to be an efficient despin device, this would be a very good approximation, but from our previous result, we see that correction for this difference can be made with no more difficulty by simply replacing m_1 by the reduced mass $m_1(1 - m_1/M)$ in Eq. (14).§

The instantaneous moment of inertia about the center of mass of the satellite and yo-yo system follows from Eq. (11) as

$$I = I_0 + I_1 + m_1(1 - m_1/M)r_1^2 \quad (15)$$

If we compare this expression with the well-known parallel axes theorem for the moment of inertia, again, the meaning of the reduced mass is clear.

The concept of a reduced mass used in a slightly different meaning is well known in the classical two-body problem (see, for instance, Ref. 4). Notice, however, when there is more than one auxiliary body in a satellite, cross-product terms between the bodies appear in Eqs. (4) and (6), and the simplicity provided by the concept of a reduced mass does not exist.

5. Discussion

The representation of the kinetic energy and angular momentum about the variable satellite center of mass in Eqs. (4) and (6) is valid not only for a satellite vehicle but also for any system of particles and rigid bodies. The relative position and velocity vectors can be those referred to any one of the bodies, whichever one is the most convenient. The advantages of these representations are as follows:

1) It is not necessary to calculate explicitly the location of the variable center of mass.

2) These representations involve only the position and velocities relative to the satellite main body center of mass. This is convenient because a) these relative positions and velocities are usually either those specified, or those which can be expressed in relatively simple analytic forms; and b) the motion of the center of mass of the satellite main body does not appear in the equations.

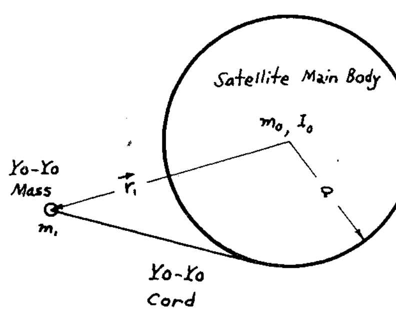

Fig. 1 Top view of yo-yo despin device for a satellite.

§ No correction is made here for the finite moment of inertia of the yo-yo mass about its own center of mass even though the procedure is just as straightforward.

It is expected that the present result should find application to satellite attitude stabilizing systems based on the principle of using small masses to execute prescribed motions with respect to the satellite main body.

References

[1] Leondes, C. T. (ed.), *Guidance and Control of Aerospace Vehicles* (McGraw-Hill Book Co., Inc., New York, 1963), Chap. 8.

[2] Roberson, R. E., "Attitude control of satellite and space vehicles," Advan. Space Sci. **2**, 351–436 (1960).

[3] Thomson, W. T., *Introduction to Space Dynamics* (John Wiley and Sons, Inc., New York, 1961), pp. 208–211.

[4] Goldstein, H., *Classical Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1950), p. 59.

Linear Feedback Solutions for Minimum Effort Interception, Rendezvous, and Soft Landing

Arthur E. Bryson Jr.*
Harvard University, Cambridge, Mass.

AND

Douglas Aircraft Company, Inc., Santa Monica, Calif.

Statement of General Problem

THERE are two moving bodies in vacuo, one the pursuing body (subscript P) and the other a target body (subscript T). The equations of motion for the two bodies are

$$\dot{\mathbf{v}}_P = \mathbf{a}_P + \mathbf{u} \qquad \dot{\mathbf{r}}_P = \mathbf{v}_P \quad (1)$$

$$\dot{\mathbf{v}}_T = \mathbf{a}_T \qquad \dot{\mathbf{r}}_T = \mathbf{a}_T \quad (2)$$

where $\mathbf{v}_P$, $\mathbf{v}_T$ = velocity vectors, $\mathbf{r}_P$, $\mathbf{r}_T$ = position vectors, $\mathbf{a}_T$ = acceleration vectors due to external forces, $\mathbf{u}$ = acceleration vector due to control force, pursuing body, $(\dot{\ }) = d(\ )/dt$. The relative motion is described by

$$\dot{\mathbf{v}} = \mathbf{a} + \mathbf{u} \qquad \dot{\mathbf{r}} = \mathbf{v} \quad (3)$$

where $\mathbf{v} = \mathbf{v}_P - \mathbf{v}_T$, $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_T$, $\mathbf{a} = \mathbf{a}_P - \mathbf{a}_T$. The problem is to find $\mathbf{u}(t)$ to minimize

$$J = \frac{1}{2}\,[c_v\mathbf{v}\cdot\mathbf{v} + c_r\mathbf{r}\cdot\mathbf{r}]_{t=T} + \frac{1}{2}\int_{t_0}^{T}\mathbf{u}\cdot\mathbf{u}dt \quad (4)$$

where T = terminal time, t_0 = initial time, c_v, c_r are scalar constants. The Hamiltonian of the problem is therefore

$$H = (\mathbf{u}\cdot\mathbf{u})/2 + \boldsymbol{\lambda}_v\cdot(\mathbf{a} + \mathbf{u}) + \boldsymbol{\lambda}_r\cdot\mathbf{v} \quad (5)$$

and the influence vectors are determined by

$$\dot{\boldsymbol{\lambda}}_v = -\partial H/\partial\mathbf{v} = -\boldsymbol{\lambda}_r \quad (6)$$

$$\dot{\boldsymbol{\lambda}}_r = -\partial H/\partial\mathbf{r} = 0 \quad (7)$$

where it was assumed that $\mathbf{a}$ was constant. If both pursuer and target are being acted upon by the same (constant) gravitational force per unit mass, then $\mathbf{a} = 0$. If the target is a fixed point, as in the soft-landing problem, then $\mathbf{a}_T = 0$ and $\mathbf{a} = \mathbf{a}_P = \mathbf{g}$ is the gravitational force per unit mass. Even though $\mathbf{g}$ is never truly constant, the approximation of treating it as constant is quite good for short flight paths.

The optimality condition is simply

$$\partial H/\partial\mathbf{u} = \mathbf{u} + \boldsymbol{\lambda}_v = 0 \rightarrow \mathbf{u} = -\boldsymbol{\lambda}_v \quad (8)$$

Received October 9, 1964; revision received April 14, 1965.

* Professor, Division of Engineering and Applied Physics, and Consultant, respectively. Associate Fellow Member AIAA.

This latter result states that the control acceleration should be in the opposite direction to the velocity influence vector $\boldsymbol{\lambda}_v$, an intuitively obvious result. Equations (6) and (7) are easily integrated:

$$\boldsymbol{\lambda}_r = \text{constant vector} \qquad \boldsymbol{\lambda}_v = -(t - t_0)\boldsymbol{\lambda}_r + \boldsymbol{\lambda}_{v_0} \quad (9)$$

where

$$\boldsymbol{\lambda}_{v_0} = \boldsymbol{\lambda}_v(t_0) \quad (10)$$

Thus, from (8) and (10), the optimal control acceleration is a linear function of time. The boundary conditions on $\boldsymbol{\lambda}_r$ and $\boldsymbol{\lambda}_v$ are

$$\boldsymbol{\lambda}_v(T) = c_v\mathbf{v}(T) \quad (11)$$

$$\boldsymbol{\lambda}_r(T) = c_r\mathbf{r}(T) \quad (12)$$

Summarizing, we have the boundary value problem

$$\dot{\mathbf{v}} = \mathbf{a} + (t - t_0)\boldsymbol{\lambda}_r - \boldsymbol{\lambda}_{v_0} \quad (13)$$

$$\dot{\mathbf{r}} = \mathbf{v} \quad (14)$$

with initial conditions $\mathbf{v}(t_0)$, $\mathbf{r}(t_0)$, and the final conditions

$$-(T - t_0)\boldsymbol{\lambda}_{r_0} + \boldsymbol{\lambda}_{v_0} = c_v\mathbf{v}(T) \quad (15)$$

$$\boldsymbol{\lambda}_{r_0} = c_r\mathbf{r}(T) \quad (16)$$

This is a straightforward boundary value problem, and the required values of $\boldsymbol{\lambda}_r$ and $\boldsymbol{\lambda}_{v_0}$ are found to be

$$\begin{bmatrix}\boldsymbol{\lambda}_{v_0}\\ \boldsymbol{\lambda}_r\end{bmatrix} = \frac{\begin{bmatrix} c_r(T - t_0)^3/6 - 1, & -(T - t_0)[c_v(T - t_0)/2 + 1)] \\ c_r(T - t_0)^2/2, & -c_v(T - t_0) - 1\end{bmatrix}\begin{bmatrix} c_v[\mathbf{a}(T - t_0) + \mathbf{v}_0] \\ c_r[\mathbf{a}(T - t_0)^2/2 + \mathbf{v}_0(T - t_0) + \mathbf{r}_0]\end{bmatrix}}{[c_v(T - t_0) + 1][c_r(T - t_0)^3/6 - 1] - c_r(T - t_0)^3[c_v(T - t_0)/2 + 1/2]} \quad \begin{matrix}(17)\\(18)\end{matrix}$$

From (8) and (10), the optimal control acceleration is given by

$$\mathbf{u} = -\boldsymbol{\lambda}_{v_0} + \boldsymbol{\lambda}_r(t - t_0) \quad (19)$$

Equation (19) may be interpreted as a "sampled-data guidance law" where t_0 was the time of the latest sample. If $\mathbf{v}$ and $\mathbf{r}$ are measured continuously, then t_0 and t are identical, and (19) becomes a "continuous feedback guidance law."

Terminal Velocity Control

For this case we set the target acceleration to zero ($\mathbf{a}_T = 0$) so that $\mathbf{a}_P = \mathbf{g} = \text{const.}$ The initial relative velocity is simply $\mathbf{v}_{P_0} - \mathbf{v}_T$ where $\mathbf{v}_T$ is the terminal velocity required. The final position is not specified and so we put $c_r = 0$ in the performance index (4). As a result, Eqs. (17) and (18) simplify greatly:

$$\left.\begin{aligned}\boldsymbol{\lambda}_{v_0} &\rightarrow \frac{\mathbf{g}(T - t_0) + \mathbf{v}_{P_0} - \mathbf{v}_T}{(T - t_0) + 1/c_v}\\ \boldsymbol{\lambda}_{r_0} &\rightarrow 0\end{aligned}\right\} \text{as} \quad c_r \rightarrow 0 \quad (20)$$

As the constant c_v is increased, the final velocity will be closer to $\mathbf{v}_T$, but, at the expense of greater control effort,

$$\int_{t_0}^{T} u^2 dt$$

In the limit, as $c_v \rightarrow \infty$, the control becomes

$$\mathbf{u} \rightarrow -g + (\mathbf{v}_T - \mathbf{v}_{P_0})/(T - t_0) \quad \text{as} \quad c_v \rightarrow \infty \quad (21)$$

Since t_0 is any initial time, this may be interpreted as follows: $\mathbf{u}$ must balance off $\mathbf{g}$ and make up the velocity difference $\mathbf{v}_T - \mathbf{v}_P$ by a constant additional acceleration $(\mathbf{v}_T - \mathbf{v}_P)/(T - t)$. This is called "velocity-to-be-gained" navigation; it may be interpreted as a continuous feedback guidance law ($t_0 \equiv t$):

$$\dot{\mathbf{v}}_P = \mathbf{g} + [\mathbf{v}_T - g(T - t) - \mathbf{v}_P]/(T - t + 1/c_v) \quad (22)$$

Interception of a Target in Free Fall

For interception of a target in free fall, $\mathbf{a}_T = \mathbf{a}_P$, and so $\mathbf{a} = 0$. The terminal velocity is not specified, so we put $c_v = 0$ in the performance index (4). Again Eqs. (17) and (18) simplify:

$$\left.\begin{aligned}\boldsymbol{\lambda}_{v_0} &\rightarrow \frac{(T - t_0)[\mathbf{v}_0(T - t_0) + \mathbf{r}_0]}{(T - t_0)^3/3 + 1/c_r} \quad (23)\\ \boldsymbol{\lambda}_{r_0} &\rightarrow \frac{\mathbf{v}_0(T - t_0) + \mathbf{r}_0}{(T - t_0)^3/3 + 1/c_r} \quad (24)\end{aligned}\right\} \text{as} \quad c_v \rightarrow 0$$

From (8) and (10), the control law becomes

$$\mathbf{u} = \{-(T - t)[\mathbf{v}_0(T - t_0) + \mathbf{r}_0]\}/[(T - t_0)^3/3 + 1/c_r] \quad (25)$$

which may be interpreted as a sampled-data feedback law where the latest sample was at t_0. If the relative velocity $\mathbf{v}$ and the relative position $\mathbf{r}$ are measured continuously (instead of just at t_0), then $\mathbf{v}_0$, $\mathbf{r}_0$, and t_0 may be interpreted as $\mathbf{v}$, $\mathbf{r}$, and t, and (25) becomes

$$\mathbf{u} \rightarrow \{(T - t)[\mathbf{v}(T - t) + \mathbf{r}]\}/[\tfrac{1}{3}(T - t)^3 + 1/c_r] \quad (26)$$

which is a continuous feedback guidance law. Let x be the lateral displacement from a nominal line-of-sight (Fig. 1). Then

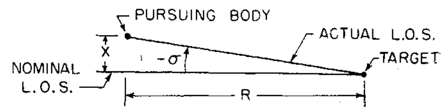

Fig. 1 Geometry for intercept case.

$$x = -R\sigma \quad (27)$$

where R is the range and σ is the angle between the actual and the nominal lines-of-sight. Thus

$$\dot{x} = -R\dot{\sigma} - \dot{R}\sigma \quad (28)$$

and substituting (27) and (28) into a lateral component of (26) gives

$$u_{\perp} = + \frac{(T - t)[(T - t)(R\dot{\sigma} + \dot{R}\sigma) + R\sigma]}{\tfrac{1}{3}(T - t)^3 + 1/c_r} \quad (29)$$

If we approximate the closing velocity V_c as constant, then

$$R \cong V_c(T - t) \quad (30)$$

and the lateral acceleration law (29) becomes

$$u_{\perp} = 3V_c\dot{\sigma}/[1 + 3/c_r(T - t)^3] \quad (31)$$

As the constant c_r is increased, it is easy to show that the final miss distance becomes smaller but at the expense of greater effort

$$\int_{t_0}^{T} u^2 dt$$

If we let $c_r \rightarrow \infty$, i.e., if we insist on zero terminal miss, then (31) becomes

$$u_{\perp} = 3V_c\dot{\sigma} \quad (32)$$

which is "proportional navigation" with a gain of 3.

Rendezvous with a Target in Free Fall

For rendezvous, $\mathbf{a} = 0$, and we wish to make the terminal relative velocity and terminal relative position very small. If we go the limit and put $c_r \rightarrow \infty$ and $c_v \rightarrow \infty$ simultaneously, then (17) and (18) become

$$\begin{bmatrix}\boldsymbol{\lambda}_{v_0}\\ \boldsymbol{\lambda}_r\end{bmatrix} \rightarrow \frac{\begin{bmatrix}(T - t_0)^3/6, & -(T - t_0)^2/2\\ (T - t_0)^2/2, & -(T - t_0)\end{bmatrix}\begin{bmatrix}\mathbf{v}_0\\ \mathbf{v}_0(T - t_0) + \mathbf{r}_0\end{bmatrix}}{-\tfrac{1}{12}(T - t_0)^4}$$

$$\text{as} \quad c_r, c_v \rightarrow \infty \quad (33)$$

or

$$\lambda_{v_0} = [4/(T - t_0)]\mathbf{v}_0 + [6/(T - t_0)^2]\mathbf{r}_0$$
$$\lambda_r = [6/(T - t_0)^2]\mathbf{v}_0 + [12/(T - t_0)^3]\mathbf{r}_0 \tag{34}$$

From (8) and (10), these relations give the guidance law

$$\mathbf{u} = -[4/(T - t)]\mathbf{v} - [6/(T - t)^2]\mathbf{r} \tag{35}$$

where, as in (26), we have interpreted t_0, $\mathbf{v}_0$, and $\mathbf{r}_0$ as t, $\mathbf{v}$, and $\mathbf{r}$.

Using the nomenclature and assumptions of (27–32), the acceleration perpendicular to the nominal line-of-sight is given as

$$u_\perp = 4V_c\dot{\sigma} + 2V_c\sigma/(T - t) \tag{36}$$

which is a modified form of proportional navigation.

Soft Landing

For soft landing $\mathbf{a}_T = 0$, $\mathbf{a}_P = \mathbf{g}$ so that $\mathbf{a} = \mathbf{g}$, and again, as in rendezvous, we wish to make the terminal relative velocity and position very small. If we go to the limit and put $c_r \to \infty$ and $c_v \to \infty$, we obtain almost the same guidance law as (35), the only difference being the additional term $\mathbf{g}$:

$$\mathbf{u} = -\mathbf{g} - [4/(T - t)]\mathbf{v} - [6/(T - t)^2]\mathbf{r} \tag{37}$$

References

[1] Bryson, A. E. and Denham, W. F., "Multivariable terminal control for minimum mean square deviation from a nominal path," *Proceedings of Symposium on Vehicle Systems Optimization* (Institute of Aerospace Sciences, New York, N. Y., 1962).

[2] Bryson, Arthur E. and Denham, Walter F., "Guidance scheme for supercircular re-entry of a lifting vehicle," ARS J. **32**, 894–898 (1962).

[3] Bryson, A. E. and Ho, Y. C., summer course notes on optimization of dynamic systems, Harvard Univ. (August 1963).

Self-Contained Preliminary Orbit Determination from Angular Measurements

Robert H. Gersten* and Z. E. Schwarzbein†
Aerospace Corporation, El Segundo, Calif.

Introduction

In three previous papers[1–3] the authors have established mathematical models for self-contained preliminary orbit determination (with all of the computations and measurements made on board). These models utilize measurements of the local vertical and stellar observations to determine the orientation of the orbit plane in inertial space and the increment in true anomaly between the first and succeeding measurements. The remaining two-body orbital elements result from additional measurements of altitude, radial velocity, and/or time rate of change of time anomaly. However, in practice, these latter quantities may not be obtained readily with existing onboard sensors. This difficulty is alleviated in the present paper, which extends the previous analyses to consider preliminary orbit determination from four sequential determinations of unit vectors in the radial direction, obtained via measurements of the local vertical and stellar observations and the corresponding times.

Received February 15, 1965; revision received May 10, 1965.

* Member of Technical Staff, Astrodynamics Department. Member AIAA.

† Member of Technical Staff, Astrodynamics Department; now Research Specialist, North American Aviation, Downey, Calif.

Derivation

1. *General Remarks*

The determination of a preliminary orbit from measurement of the local vertical and stellar observations can be logically divided into three parts. First it is necessary to determine the orientation of the orbit plane and the increments in true anomaly Δv_{1j} ($j = 2, 3, 4$) between the first and succeeding measurements (here the subscripts correspond to observation times). One must then compute the in-plane orbital elements. Finally, one can determine the position and velocity vectors at the first observation time and thus fix the initial two-body orbit. The first and third steps are discussed in detail in Ref. 1 and will not be treated here. Although the requirement for four unit vectors appears to provide a redundancy in observational data, it should be noted that the only useful quantities for determining the in-plane orbital elements by this method are the increments in true anomaly. Thus, since these increments in true anomaly are obtained from the dot products of the unit vectors, four unit vectors are required to define the three increments in true anomaly necessary to determine the three in-plane orbital elements.

2. *Determination of in-plane orbital elements*

This section is concerned with the determination, from three sequential values of Δv_{1j}, of the parameter p, (semilatus rectum), orbital eccentricity e, and true anomaly at the initial observation time v_1. Two alternate formulations are presented. The first is an exact two-body determination, whereas the second is valid only for relatively low eccentricity orbits (say $e < 0.2$). The lack of generality of the second approach is compensated by considerable simplification in the formulation and a corresponding decrease in onboard computation requirements.

Exact method: The time of flight between the first and jth observations for elliptical, parabolic, and hyperbolic orbits are, respectively,[4]

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \frac{1}{1 - e^2} \left\{ \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(1 - e^2)^{1/2}} \times \tan^{-1}\left[\left(\frac{1 - e}{1 + e}\right)^{1/2} \tan\frac{v}{2}\right]\right\}\Bigg|_{v_1}^{v_1 + \Delta v_{1j}} \tag{1a}$$

$$t_j - t_1 = \frac{1}{2}\left(\frac{p^3}{\mu}\right)^{1/2} \tan\frac{v}{2}\left(1 + \frac{1}{3}\tan^2\frac{v}{2}\right)\Bigg|_{v_1}^{v_1 + \Delta v_{1j}} \tag{1b}$$

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \frac{1}{1 - e^2} \left\{ \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(e^2 - 1)^{1/2}} \tanh^{-1}\left[\left(\frac{e - 1}{e + 1}\right)^{1/2} \tan\frac{v}{2}\right]\right\}\Bigg|_{v_1}^{v_1 + \Delta v_1} \tag{1c}$$

$$j = 2, 3, 4$$

where t is the time, and μ is the product of the universal gravitational constant and the mass of the principal attracting body. It should be noted in using Eq. (1a) that the $\tan^{-1}$ term lies in the same quadrant as $v/2$. Let us define for elliptical, parabolic, and hyperbolic orbits, respectively,

$$f(e, v) = \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(1 - e^2)^{1/2}} \tan^{-1} \times \left[\left(\frac{1 - e}{1 + e}\right)^{1/2} \tan\frac{v}{2}\right] \tag{2a}$$

$$f(e, v) = \tan(v/2)\left[1 + \frac{1}{3}\tan^2(v/2)\right] \tag{2b}$$